General framework for the behaviour of continuously observed open quantum systems

2002 J. Phys. A: Math. Gen. 35 565

(http://iopscience.iop.org/0305-4470/35/3/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 565–588 PII: S0305-4470(02)28304-3

General framework for the behaviour of continuously observed open quantum systems

Ole E Barndorff-Nielsen[1] **and Elena R Loubenets**[2]

[1] MaPhySto, University of Aarhus, DK-8000 Aarhus, Denmark
[2] Moscow State Institute of Electronics and Mathematics, Technical University, Trekhsvyatitelskii Per. 3/12, Moscow 109028, Russia

E-mail: oebn@imf.au.dk and erl@erl.msk.ru

Received 24 August 2001
Published 11 January 2002
Online at stacks.iop.org/JPhysA/35/565

Abstract

We develop the general quantum stochastic approach for the description of quantum measurements continuous in time. The framework that we introduce encompasses the various particular models for continuous-time measurements considered previously in the physical and the mathematical literature.

PACS numbers: 03.65.Yz, 02.50.−r, 03.65.Ta

1. Introduction

In recent years, the problem of describing the behaviour of a quantum system continuously observed in time has been the subject of intensive investigations in the physical and the mathematical literature. This type of behaviour is generally not reversible in time and hence, in particular, cannot be described by the Schrödinger equation whose solutions are reversible. The present strong interest in this fundamental problem is to a large extent caused by the rapid development of experimental techniques, where experiments involving continuous-time (i.e. continuous in time) 'monitoring' of a quantum system have become possible [19–21, 31, 41].

The present paper develops, in the context of continuous-time monitoring, the general approach for the description of quantum measurements, formulated in [36, 37]. The general framework that we introduce encompasses the various particular models for continuous-time measurements, considered previously in the physical and mathematical literature.

This concerns:

- The Markovian models in the mathematical physics literature [1–10, 27, 30], formulated either in terms of stochastic differential equations or in terms of semigroups of probability operators or in terms of the instrumental processes with independent increments. In fact, the stochastic equations used in all of these models generalize the quantum filtering equation which was derived in the quantum stochastic calculus modelling framework.

This modelling framework satisfies the principles of nondemolition observation ([7–10] and references therein). It was, however, shown in [35] that the case of continuous-time indirect nondemolition measurement can be considered in the more general quantum theory setting, which is not based on the use of the essentially Markovian measurement model of quantum stochastic calculus.

- The models of continuous-time observation in the physical literature [11, 13, 14, 17, 22–25, 34, 38, 39, 44, 47–51], including those in quantum optics. The derivation of stochastic equations in all these models is based mostly on unravelling the master equation of Lindblad type [33] (cf, for example, [39, 50, 51]) or on the phenomenological introduction of non-Markovian quantum trajectories [47–49].

As a prerequisite for our results in the main part of the paper, we review in section 2 the main concepts of the operational approach (section 2.1) and the main ideas of the quantum stochastic approach (QSA) (section 2.2) for the description of quantum measurements.

Sections 3–8 then develop the formalism for the description of continuous-time measurements from the general viewpoint of the QSA, formulated in [36, 37]. The general scene is set in section 3. Section 4 introduces the notion of a posterior pure state trajectory and gives its probabilistic treatment. The special case of Markov evolution is treated in section 5. Section 6 establishes the notion of a measuring model of continuous-time direct quantum measurement and sections 7 and 8 address the questions of continuous-time nondemolition measurement. Finally, section 9 consists of concluding remarks.

2. Basic representations of quantum measurements

This section reviews both the operational approach and the quantum stochastic approach for the description of quantum measurements.

By a quantum measurement we mean a physical experiment on a quantum system which, resulting in the observation in the classical world of an outcome that (to some degree) characterizes the quantum system, may cause a change in the state of the quantum system, but not the quantum system's destruction.

We distinguish between *direct* and *indirect* quantum measurements. A direct quantum measurement corresponds to a measurement situation where we have to describe the direct interaction between the measuring device and the observed quantum system, while in case of an indirect measurement, a direct measurement is made of some other quantum system, entangled with the one considered.

The term 'generalized measurement', as usual, corresponds to the measurement situation with outcomes of the most general nature possible under a quantum measurement.

Let a quantum system S, described in terms of a complex separable Hilbert space $\mathcal{H}$, interact with another system (quantum or classical). The interaction, changing the initial state ρ_0 of S into a certain new state, leaves some imprint in the classical world, the imprint being described as a point ω in some standard Borel measure space[3] $(\Omega, \mathcal{F})$. Denote the Banach space of all bounded linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Consider first the most general scheme of the *complete statistical description* of any generalized quantum measurement. This kind of description implies the knowledge of the probability distribution of different outcomes of a measurement and a statistical description of the state change of the quantum system under this measurement.

[3] A standard Borel space is a measurable space that is isomorphic to the real unit interval. In particular, any Polish space is standard Borel.

We introduce the following notation:

Let $\pi(B;\rho_0) = \text{Prob}\{\omega \in B;\rho_0\}$ be the probability that the imprint ω in the classical world belongs to a subset $B \in \mathcal{F}$.

Let $\text{Ex}\{Z;\rho_0|B\}$ be the conditional expectation of any von Neumann observable $Z = Z^*$, $Z \in \mathcal{B}(\mathcal{H})$, at the instant immediately after the measurement and conditioned on the outcome $\omega \in B$.

The posterior state (or posterior density operator) $\rho_{\text{out}}(B;\rho_0)$, of the quantum system conditioned by the imprint B in the classical world, is defined indirectly as the solution to

$$\text{Ex}\{Z;\rho_0|B\} = \text{tr}\{\rho_{\text{out}}(B;\rho_0)Z\} \tag{2.1}$$

(for arbitrary Z and B) and constitutes the statistical description of the state change of the quantum system under a measurement when only the event that ω belongs to B has been recorded (cf [42, 43, 2, 36, 37]).

The unconditional posterior state $\rho_{\text{out}}(\Omega;\rho_0)$ of the quantum system corresponds to the situation where the imprint ω in the classical world is ignored completely.

Any posterior state $\rho_{\text{out}}(B;\rho_0)$ can be described in terms of a family of statistical operators $\{\rho_{\text{out}}(\omega;\rho_0), \omega \in \Omega\}$, defined π-almost everywhere (a.e.) on Ω, and usually referred to as the *family of posterior states*. Specifically, for all $B \in \mathcal{F}$ with $\pi(B;\rho_0) \neq 0$,

$$\rho_{\text{out}}(B;\rho_0) = \frac{\int_B \rho_{\text{out}}(\omega;\rho_0)\pi(\mathrm{d}\omega;\rho_0)}{\pi(B;\rho_0)}. \tag{2.2}$$

For the unconditional posterior state $\rho_{\text{out}}(\Omega;\rho_0)$ the relation (2.2) can be considered as the usual statistical average over the posterior states $\rho_{\text{out}}(\omega;\rho_0)$ with respect to the probability distribution $\pi(\mathrm{d}\omega;\rho_0)$.

For any type (direct or indirect) of generalized quantum measurement the operational approach [15, 16, 26, 32, 42, 43, 2, 29] can be used for the most general mathematical specification of all of the above-mentioned elements of the statistical description of a measurement.

2.1. The operational description of a generalized quantum measurement

In the frame of the operational approach, the mathematical notion of a *quantum instrument* plays a central role.

Specifically, a mapping $\mathcal{N}(\cdot)[\cdot]\colon \mathcal{F} \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called a quantum instrument if $\mathcal{N}(\cdot)$ is a σ-additive measure on $(\Omega, \mathcal{F})$ with values $\mathcal{N}(B)$, $B \in \mathcal{F}$, that are normal completely positive[4] bounded linear maps $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that the following normalization relation is valid: $\mathcal{N}(\Omega)[I] = I$.

In the frame of the operational approach it is assumed that

$$\text{Ex}\{Z;\rho_0|B\} = \frac{\text{tr}\{\rho_0\mathcal{N}(B)[Z]\}}{\pi(B;\rho_0)} \qquad \forall B \in \mathcal{F}. \tag{2.3}$$

In case $Z = I$, from (2.3) it follows that the probability distribution $\pi(B;\rho_0)$ of outcomes under a measurement is given by

$$\pi(B;\rho_0) = \text{tr}\{\rho_0\mathcal{N}(B)[I]\} \qquad \forall B \in \mathcal{F}. \tag{2.4}$$

The positive σ-additive operator-valued measure $M(B) = \mathcal{N}(B)[I]$, satisfying the condition $M(\Omega) = I$, is called a *probability operator-valued measure* or a POV measure, for short.

[4] For the definitions of normality and complete positivity see, for instance, [29].

Due to (2.3), in the frame of the operational approach the posterior state $\rho_{\text{out}}(B;\rho_0)$, conditioned by the outcome $\omega \in B$ and defined by the relation (2.1), is representable as

$$\rho_{\text{out}}(B;\rho_0) = \frac{\mathcal{M}(B)[\rho_0]}{\pi(B;\rho_0)} \tag{2.5}$$

where $\mathcal{M}(B)[\cdot]$ denotes the map dual to $\mathcal{N}(B)[\cdot]$, which acts on the Banach space $\mathcal{T}(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ and is defined by

$$\text{tr}\{\kappa\mathcal{N}(B)[Y]\} = \text{tr}\{\mathcal{M}(B)[\kappa]Y\} \tag{2.6}$$

for arbitrary $Y \in \mathcal{B}(\mathcal{H}), \kappa \in \mathcal{T}(\mathcal{H})$. Since $\mathcal{N}(\Omega)[I] = I$, it follows from (2.6) that $\text{tr}\{\mathcal{M}(\Omega)[\rho]\} = 1$ for any density operator $\rho \in \mathcal{T}(\mathcal{H})$. We follow the terminology of [29] and refer to $\mathcal{M}(\cdot)[\cdot]$ as a quantum instrument associated with the quantum instrument $\mathcal{N}(\cdot)[\cdot]$. Due to (2.5) we also have

$$\pi(B;\rho_0) = \text{tr}\{\mathcal{M}(B)[\rho_0]\} \qquad \forall B \in \mathcal{F}. \tag{2.7}$$

For any initial state ρ_0 of a quantum system, the family of posterior states $\{\rho_{\text{out}}(\omega,\rho_0), \omega \in \Omega\}$ always exists [42, 43, 2] and is defined uniquely, π-almost everywhere, by the relation:

$$\int_B \text{tr}\{\rho_{\text{out}}(\omega;\rho_0)Y\}\pi(\mathrm{d}\omega,\rho_0) = \text{tr}\{\rho_0\mathcal{N}(B)[Y]\} \tag{2.8}$$

for all $Y \in \mathcal{B}(\mathcal{H}), \forall B \in \mathcal{F}$. From (2.2) and (2.5) we have, in particular,

$$\rho_{\text{out}}(\omega;\rho_0) = \frac{\mathrm{d}\mathcal{M}(\cdot)[\rho_0]}{\mathrm{d}\pi(\cdot;\rho_0)} \tag{2.9}$$

that is, the posterior state $\rho_{\text{out}}(\omega;\rho_0)$ is a density of the measure $\mathcal{M}(\cdot)[\rho_0]$ with respect to the probability scalar measure $\pi(\cdot;\rho_0)$. Further, from (2.5) it follows that the unconditional posterior state is given by

$$\rho_{\text{out}}(\Omega;\rho_0) = \mathcal{M}(\Omega)[\rho_0]. \tag{2.10}$$

It was proved in [36] that for any quantum instrument there exist:

- a positive finite scalar measure $\nu(\cdot)$ on $(\Omega, \mathcal{F})$;
- a positive integer $N_0 \leqslant \infty$;
- a dimension function $N(\cdot)$, defined ν-almost everywhere on Ω, with values being positive integers $N(\omega) \leqslant \infty$;
- positive numbers α_i, summing up to one $\sum_{i=1}^{N_0}\alpha_i = 1$;
- a family $\{W_{in}: i = 1,\ldots,N_0; n = 1,\ldots,l\}$ (with l being equal to ν-sup$\{N(\omega), \omega \in \Omega\}$) of bounded linear operators $W_{in}\colon \mathcal{H} \to \mathcal{L}_2(\Omega,\nu;\mathcal{H})$, satisfying for $\forall f, g \in \mathcal{H}$ the orthonormality relation

$$\int_\Omega \sum_{n=1}^{N(\omega)} \langle (W_{jn}f)(\omega), (W_{in}g)(\omega)\rangle \nu(\mathrm{d}\omega) = \langle f, g\rangle \delta_{ji} \tag{2.11}$$

such that for $\forall B \in \mathcal{F}, \forall Y \in \mathcal{B}(\mathcal{H})$ and $\forall f, g \in \mathcal{H}$ the following integral representation for a quantum instrument is valid:

$$\langle f, \mathcal{N}(B)[Y]g\rangle = \sum_{i=1}^{N_0}\alpha_i \int_B \sum_{n=1}^{N(\omega)} \langle (W_{in}f)(\omega), Y(W_{in}g)(\omega)\rangle \nu(\mathrm{d}\omega). \tag{2.12}$$

The integral representation (2.12) is, in general, different and more detailed than the representations available in the mathematical and physical literature [28, 45, 51]. The latter are similar to the Stinespring–Kraus representation for a completely positive map on $\mathcal{B}(\mathcal{H})$ (cf, for example, [29]). The most essential difference is due to the orthonormality relation (2.11), which is not present in the Stinespring–Kraus like representations of a quantum instrument [5, 28, 29, 45, 51]. Moreover, since the two different types of indices i, n enter the orthonormality relation (2.11) in quite different manners, the double indexing in (2.12) cannot, in general, be presented as a single one without loss of the natural structure of an orthonormality relation (see [36] for further discussion).

We would like to emphasize here that having the elements of an integral representation of an instrument, one can construct (due to the definite transformation rule (see [36])) a plenitude of other integral representations of the same instrument with different families of operators $\{W'_{in}\}$ and different scalar measures ν', the latter being, however, of the same type: $[\nu'] = [\nu]$.

The operational approach, while essential for the formalization of the statistical description of any generalized quantum measurement, does not, in general, specify a possible random behaviour of the quantum system under a single measurement. In other words, the operational approach, in general, does not give the possibility to include into consideration the description of the stochastic, irreversible in time behaviour of a quantum system under a single measurement, depending on an outcome ω in the classical world. The description of such stochastic behaviour of a quantum system means the specification of a probabilistic transition law governing the change from the initial state of the quantum system to a final one under a single quantum measurement. We refer to this kind of description of a quantum measurement as a *complete stochastic description*.

The operational approach also does not distinguish between direct and indirect measurements.

In this connection we would like to emphasize that in quantum theory any physically based problem must be formulated in unitarily equivalent terms and the results of its consideration must not be dependent either on the choice of a special representation picture (Schrödinger, Heisenberg or interaction) or on the choice of basis in the Hilbert space. Moreover, generally the description of any direct quantum measurement cannot simply be reduced to the quantum theory description of a measuring process, as it is now usually considered in the mathematical and physical literature. For this kind of measurement situation we cannot specify definitely either the interaction, or the quantum state of a measuring device environment, or describe a measuring device in quantum theory terms alone. In fact, under such a scheme, the description of a direct quantum measurement is simply transferred to the description of a direct measurement of some observable of an environment of a measuring device.

We recall that for the case of discrete outcomes, the original von Neumann approach [40] describes *specifically a direct quantum measurement* and gives both a complete statistical description and a complete stochastic description of this measurement. Specifically, if the initial state ρ_0 of a quantum system is pure, that is, $\rho_0 = |\psi_0\rangle\langle\psi_0|$, and if under a single measurement the outcome λ_j is observed, then in the frame of the von Neumann approach the quantum system 'jumps' with *certainty* to the posterior pure state

$$\frac{P_j|\psi_0\rangle\langle\psi_0|P_j}{\|P_j\psi_0\|} \tag{2.13}$$

where P_j is the projection, corresponding to the observed eigenvalue λ_j of the observable $Z = \sum_j \lambda_j P_j$. The probability μ_j of the outcome λ_j is given by

$$\mu_j = \|P_j\psi_0\|^2. \tag{2.14}$$

An approach giving both a complete statistical and a complete stochastic description of a *direct* quantum measurement with outcomes of the most general possible nature was introduced in [36, 37]. This approach is called *quantum stochastic* and we refer to it as QSA.

2.2. Quantum stochastic approach

It was shown in [36] that any generalized direct quantum measurement (section 2.1) can be described in terms of certain scalar measures on a standard Borel space $(\Omega, \mathcal{F})$ and associated stochastic evolution operators, describing the stochastic evolution of the quantum system in the Hilbert space $\mathcal{H}$ conditioned by the observed outcome ω. We refer to the collection of these quantities as a *quantum stochastic representation*, or QSR, of a generalized direct quantum measurement. For simplicity, we consider below only quantum stochastic representations for which the quantum stochastic evolution operators are bounded.

From the point of view of the operational approach, the QSA specifies, in particular, the type of quantum instrument, corresponding to the description of a generalized direct quantum measurement.

In particular, it was shown in [36] that any *generalized direct quantum measurement* can be interpreted to correspond to an *invariant class of unitarily equivalent measuring processes* (statistical realizations). For an invariant class of measuring processes the elements of the integral representation (2.12) of the corresponding instrument are the same for all measuring processes from this class and are given only through the *unitary invariants* of the measuring process. The special form of this integral representation of an instrument, corresponding to the invariant class, is called *quantum stochastic*.

According to the QSA, for every generalized direct quantum measurement there exists a unique quantum stochastic representation of a measurement, giving a complete statistical and stochastic description of this measurement, in a precisely specified sense.

Specifically, by a *quantum stochastic representation* (QSR), we mean a collection

$$Q = \{\{q_{ji}(\omega)\nu(\mathrm{d}\omega)\}, \{V_i(\omega)\}, \{\alpha_i\}\} \tag{2.15}$$

consisting of three families of elements where:

- $q_{ji}(\omega)\nu(\mathrm{d}\omega), i, j = 1, \ldots, N_0$; $N_0 \leqslant \infty$ are complex scalar measures on a standard Borel space $(\Omega, \mathcal{F})$, absolutely continuous with respect to a finite positive scalar measure $\nu(\cdot)$, with $q_{ii}(\omega) \geqslant 0$ and satisfying the orthonormality relation
$$\int_\Omega q_{ji}(\omega)\nu(\mathrm{d}\omega) = \delta_{ji}; \tag{2.16}$$
- the $\alpha_i, i = 1, \ldots, N_0$ constitute a finite or countable sequence of positive numbers that sum to 1;
- each $V_i(\omega), i = 1, \ldots, N_0$ is a ν-measurable operator-valued function with values being linear bounded operators on $\mathcal{H}$, satisfying the orthonormality relation
$$\int_\Omega V_j^*(\omega)V_i(\omega)q_{ji}(\omega)\nu(\mathrm{d}\omega) = \delta_{ij}I \tag{2.17}$$
and such that, for any $B \in \mathcal{F}$ and any index i,
$$\int_{\omega\in B} V_i(\omega)q_{ii}(\omega)\nu(\mathrm{d}\omega) \in \mathcal{B}(\mathcal{H}). \tag{2.18}$$

We let

$$\nu_i(\mathrm{d}\omega) = q_{ii}(\omega)\nu(\mathrm{d}\omega) \tag{2.19}$$

$$\nu_0(\mathrm{d}\omega) = \sum_i \alpha_i\nu_i(\mathrm{d}\omega) \tag{2.20}$$

and refer to these as the *input probability scalar measures*.

In the case the index set for i consists of one element only, we drop the index and assume that the probability density q_{11} is identically 1, implying that $\nu(\cdot)$ is a probability measure, and we then say that the QSR is *simple*.

The ν-measurable operator-valued functions $V_i(\omega)$, having the properties (2.17) and (2.18) are called in [36] *quantum stochastic evolution operators*.

Consider, in general, the statistical and stochastic description of a quantum measurement, represented by a QSR.

The quantum instrument, corresponding to a direct quantum measurement, which is determined by the quantum stochastic representation Q, is given, for all $B \in \mathcal{F}$ and all $y \in \mathcal{B}(\mathcal{H})$, by

$$\mathcal{N}(B)[Y] = \sum_i \alpha_i \mathcal{N}_i(B)[Y] \tag{2.21}$$

with

$$\mathcal{N}_i(B)[Y] = \int_B V_i^*(\omega) Y V_i(\omega) \nu_i(\mathrm{d}\omega). \tag{2.22}$$

The probability scalar measure $\pi(\mathrm{d}\omega; \rho_0)$ on Ω, defined by (2.4), and the family of un-normalized posterior states $\eta_{\mathrm{out}}(\omega; \rho_0)$ on $\mathcal{H}$ are presented by the following specifications

$$\pi(\mathrm{d}\omega; \rho_0) = \sum_i \alpha_i \operatorname{tr}\{V_i(\omega)\rho_0 V_i^*(\omega)\}\nu_i(\mathrm{d}\omega) \tag{2.23}$$

$$\eta_{\mathrm{out}}(\omega; \rho_0) = \sum_i \alpha_i V_i(\omega)\rho_0 V_i^*(\omega) q_{ii}(\omega). \tag{2.24}$$

Introducing for every index $i = 1, \ldots, N_0$ the un-normalized posterior state

$$\eta_{\mathrm{out}}^{(i)}(\omega; \rho_0) = V_i(\omega)\rho_0 V_i^*(\omega) \tag{2.25}$$

we present the un-normalized posterior states (2.24) and the probability scalar measure (2.23) as

$$\eta_{\mathrm{out}}(\omega; \rho_0) = \sum_i \alpha_i q_{ii}(\omega) \eta_{\mathrm{out}}^{(i)}(\omega; \rho_0) \tag{2.26}$$

and

$$\pi(\mathrm{d}\omega; \rho_0) = \sum_i \alpha_i \pi_i(\mathrm{d}\omega; \rho_0) \tag{2.27}$$

with

$$\pi_i(\mathrm{d}\omega; \rho_0) = \operatorname{tr}\left\{\eta_{\mathrm{out}}^{(i)}(\omega; \rho_0)\right\} \nu_i(\mathrm{d}\omega). \tag{2.28}$$

The probability scalar measures $\pi_i(\cdot; \rho_0)$ and $\pi(\cdot; \rho_0)$ are called *output probability measures*.

Due to (2.8), (2.26) and (2.28), for the associated instrument $\mathcal{M}(\cdot)[\cdot]$ we have the following representation

$$\mathcal{M}(B)[\rho_0] = \sum_i \alpha_i \mathcal{M}_i(B)[\rho_0] \qquad \forall B \in \mathcal{F} \tag{2.29}$$

where

$$\mathcal{M}_i(B)[\rho_0] = \int_B \eta_{\mathrm{out}}^{(i)}(\omega; \rho_0)\nu_i(\mathrm{d}\omega) \tag{2.30}$$

and, consequently, for any index i the unnormalized posterior state $\eta_{\mathrm{out}}^{(i)}(\omega; \rho_0)$ can be considered as the Radon–Nikodym derivative $\frac{\mathrm{d}\mathcal{M}_i}{\mathrm{d}\nu_i}$ of the ith associated instrument $\mathcal{M}_i$ with respect to the input probability measure ν_i.

If the recorded result in the classical world is (only) that the outcome ω belongs to a certain set $B \in \mathcal{F}$ then the corresponding probability of this and the ensuing knowledge of the (normalized) posterior state of the quantum system are represented, respectively, as

$$\pi(B; \rho_0) = \int_B \pi(\mathrm{d}\omega; \rho_0) \tag{2.31}$$

and

$$\rho_{\mathrm{out}}(B; \rho_0) = \frac{\sum_i \alpha_i \int_B \eta^{(i)}_{\mathrm{out}}(\omega; \rho_0)\nu_i(\mathrm{d}\omega)}{\pi(B; \rho_0)}. \tag{2.32}$$

Due to the decompositions (2.22), (2.26) and (2.27), in the frame of the QSA $\mathcal{N}_i(\cdot)[\cdot]$, $\mathcal{M}_i(\cdot)[\cdot]$, $\eta^{(i)}_{\mathrm{out}}(\omega; \rho_0)$, $\nu_i(\mathrm{d}\omega)$ and $\pi_i(\cdot; \rho_0)$ are interpreted to present the instrument, the associated instrument, the unnormalized posterior state, the input and the output probability distributions of outcomes in the ith random transition channel of a measurement, respectively. The statistical weights of the different channels i are given by α_i, which are interpretable as probabilities.

Let the initial state of a quantum system be pure: $\rho_0 = |\psi_0\rangle\langle\psi_0|$. Due to the orthonormality relation (2.17) every pure state $V_i(\omega)\psi_0, i = 1, \ldots, N_0$, is interpreted in the frame of the QSA as a *posterior pure state outcome* in the Hilbert space $\mathcal{H}$ conditioned by the observed outcome ω and corresponding to the ith random transition channel of the quantum measurement.

For the observed outcome ω the probability of the posterior pure state outcome $V_i(\omega)\psi_0$ in $\mathcal{H}$ is given by

$$\theta_i(\omega) = \frac{\alpha_i q_{ii}(\omega)\|V_i(\omega)\psi_0\|^2}{\sum_j \alpha_j q_{jj}(\omega)\|V_j(\omega)\psi_0\|^2}. \tag{2.33}$$

The representation of the unconditional posterior state as

$$\rho_{\mathrm{out}}(\Omega; \rho_0) = \sum_i \alpha_i \int_\Omega V_i(\omega)|\psi_0\rangle\langle\psi_0|V_i^*(\omega)\nu_i(\mathrm{d}\omega) \tag{2.34}$$

is considered in the QSA as the usual statistical average over the posterior pure state outcomes $|V_i(\omega)\psi_0\rangle\langle V_i(\omega)\psi_0|, i = 1, 2, \ldots$ with respect to the input probability distribution of outcomes $\nu_i(\cdot)$ in channel i and with respect to the different channels, given with statistical weights $\alpha_i, i = 1, 2, \ldots$.

Physically, the concept of different random channels correponds, under the same outcome ω, to different underlying random quantum transitions of the environment of a measuring device, which we cannot, however, specify with certainty.

Direct measurements, on a given quantum system, described by different QSR are called *stochastic representation equivalent* provided the QSR give the same statistical and stochastic description. For example, in the frame of the QSA, the notion of a von Neumann (projective) measurement of a discrete observable $Z = \sum_j \lambda_j P_j$ corresponds to the stochastic representation equivalence class of direct measurements on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, for which the complete statistical and stochastic description is determined by the von Neumann measurement postulates [40], presented by the formulae (2.13) and (2.14).

3. Continuous-time direct measurements in the frame of QSA

We would now like to introduce the general QSR describing a continuous, over a time period $(0, T]$, direct quantum measurement. In this case the outcome ω, characterizing continuous-time observation up to the moment $0 < t \leqslant T$, is given by a record $\{x_\tau\}_{\tau\in(0,t]}$, presenting a trajectory $x_0^t = \{x_\tau\}_{\tau\in(0,t]}$ in a filtered standard Borel space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F})$, and without essential loss of generality we think of x_t as real-valued and, for simplicity, we consider the case where the measure space Ω is represented by $D(0, T]$, the space of right continuous functions with left limits, defined on $(0, T]$. In this case, for any time $t \in (0, T]$ the trajectory x_0^t is cadlag (continue a droite, limite a gauche). Further, $\mathcal{F}_\tau^t$ denotes the σ-algebra generated by $x_\tau^t = \{x_s\}_{s\in(\tau,t]}$ and we use the notation Ω_τ^t for the restriction of $D(0, T]$ to the interval $(\tau, t]$.

As discussed in section 2, under the QSA for any generalized direct quantum measurement there exists a unique QSR. Then, in particular, under a continuous-time direct quantum measurement there must exist a unique QSR, describing this special kind of generalized direct measurement. The elements of this QSR must have the time-wise properties that we describe now.

For simplicity, we consider only the case of simple QSRs. Thus, in the frame of the QSA, for any continuous-time direct quantum measurement, whose QSR is simple, at any moment of time $t \in (0, T]$ there exist:

- A unique input probability scalar measure $\nu_0^t(\cdot)$ on Ω_0^t;
- A unique family of measurable (with respect to $\mathcal{F}_t$) operator-valued functions $\{V_0^t(x_0^t)\colon x_0^t \in \Omega_0^t\}$, defined ν_0^t-almost everywhere on Ω_0^t, with values being bounded linear operators on $\mathcal{H}$ such that for any $B_0^t \in \mathcal{F}_t$

$$\int_{B_0^t} V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) \in \mathcal{B}(\mathcal{H}) \tag{3.1}$$

and the following normalization relation is valid:

$$\int_{\Omega_0^t} \left(V_0^t\left(x_0^t\right)\right)^* V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) = I. \tag{3.2}$$

From (2.22) it follows that for any continuous-time direct quantum measurement with a simple QSR at any moment of time t the instrument $\mathcal{N}_0^t(\cdot)[\cdot]$ must be represented as

$$\mathcal{N}_0^t\left(B_0^t\right)[Y] = \int_{B_0^t} \left(V_0^t\left(x_0^t\right)\right)^* Y V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) \tag{3.3}$$

for $\forall B_0^t \in \mathcal{F}_t, \forall Y \in \mathcal{B}(\mathcal{H})$, with similar time-wise notation for the associated instrument $\mathcal{M}_0^t(\cdot)[\cdot]$, the POV measure $M_0^t(\cdot)$, the output laws $\pi_0^t(\cdot;\rho_0)$ and the family of unnormalized posterior states $\{\eta_{\text{out}}^t(x_0^t;\rho_0)\colon x_0^t \in \Omega_0^t\}$, defined ν_0^t,-a.e. on Ω_0^t.

Furthermore, we must include in the specification of the QSR, describing the continuous-time direct measurement, the conditions that:

- At all moments of time until T the input probability scalar measures (describing physically the measurement situation under which the quantum system is not entangled with a measuring device) must be compatible in time;
- The output laws $\pi_0^t(\cdot;\rho_0)$ should also be compatible in time, corresponding to the compatibility in time of the POV measures $M_0^t(\cdot)$;
- We assume that for any initial pure state $\psi_0 \in \mathcal{H}$ under the continuous-time observation the posterior pure state outcome, being a trajectory in the Hilbert space $\mathcal{H}$, presented at any moment t by the quantum stochastic evolution operator as $V_0^t(x_0^t)\psi_0$, is continuous in t from the right in the norm on $\mathcal{H}$, ${\nu_0}^t$-a.e. on Ω_0^t, with the limit of $V_0^t(x_0^t)\psi_0$ as $t \downarrow 0$ being equal to ψ_0. Under this specification, the situations where the quantum system is isolated are included in our representation as a special case. In this case, for any t the quantum stochastic operator V_0^t does not depend on the event x_0^t in the classical world and

is given by a unitary operator $U(t, 0)$, strongly continuous in t for any $0 < t \leqslant T$ both from the left and the right.

Summing up all the above-mentioned points, we introduce the following time-wise specification for the elements of the simple QSR, describing a continuous-time direct quantum measurement:

- A unique collection $\left\{\nu_\tau^t(\cdot)\colon 0 \leqslant \tau < t \leqslant T\right\}$ of input probability scalar measures such that every ν_τ^t on Ω_τ^t is the restriction of the input probability scalar measure ν on Ω_0^T:

$$\nu_\tau^t\left(B_\tau^t\right) = \nu\left(\Omega_0^\tau \times B_\tau^t \times \Omega_t^T\right) \tag{3.4}$$

- A unique family $\left\{V_0^t\left(x_0^t\right)\colon x_0^t \in \Omega_0^t, 0 < t \leqslant T\right\}$ of measurable (with respect to $\mathcal{F}_t$) operator-valued functions $V_0^t(\cdot)\colon \Omega_0^t \to \mathcal{B}(\mathcal{H})$, defined ν_0^t-almost everywhere on Ω_0^t, such that, for any $0 < t \leqslant T$ and any $B_0^t \in \mathcal{F}_t$,

$$\int_{B_0^t} V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) \in \mathcal{B}(\mathcal{H}). \tag{3.5}$$

  These operator-valued functions satisfy the normalization relation

$$\int_{\Omega_0^t} \left(V_0^t\left(x_0^t\right)\right)^* V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) = I \tag{3.6}$$

  and the initial condition

$$\lim_{t\downarrow 0} \left\| V_0^t\left(x_0^t\right)\psi - \psi\right\|_{\mathcal{H}} = 0 \qquad \forall\psi \in \mathcal{H} \tag{3.7}$$

  ν_0^t-a.e. on Ω_0^t;

- A unique family $\left\{V_\tau^t\left(x_0^t\right)\colon x_0^t \in \Omega_0^t, 0 < \tau \leqslant t \leqslant T\right\}$ of measurable (with respect to $\mathcal{F}_t$) operator-valued functions $V_\tau^t\left(x_0^t\right)\colon \Omega_0^t \to \mathcal{B}(\mathcal{H})$, defined ν_0^t-almost everywhere on Ω_0^t, and such that for any $0 < \tau < t \leqslant T$ and any $B_\tau^t \in \mathcal{F}_\tau^t$, $x_0^\tau \in \Omega_0^\tau$

$$\int_{B_\tau^t} V_\tau^t\left(x_0^t\right) \nu_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right) \in \mathcal{B}(\mathcal{H}) \tag{3.8}$$

  and the following normalization relation is valid[5]

$$\int_{\Omega_\tau^t} \left(V_\tau^t\left(x_0^t\right)\right)^* V_\tau^t\left(x_0^t\right) \nu_\tau^t\left(\mathrm{d}x_\tau^t \mid x_0^\tau\right) = I. \tag{3.9}$$

  These operator-valued functions are associated with the family of operators $\left\{V_0^t\left(x_0^t\right)\right\}$ via the cocycle condition

$$V_\tau^t\left(x_0^t\right) = V_s^t\left(x_0^t\right) V_\tau^s\left(x_0^s\right) \tag{3.10}$$

  valid for any $t \in (0, T]$, $\tau \in [0, T]$, $s \in (0, T]$, $\tau \leqslant s \leqslant t$, ν_0^t-a.e. on Ω_0^t and where $V_\tau^t\left(x_0^t\right)\big|_{t=\tau} = I$. Furthermore,

$$\lim_{t\downarrow \tau} \left\| V_\tau^t\left(x_0^t\right)\psi - \psi\right\|_{\mathcal{H}} = 0 \qquad \forall\psi \in \mathcal{H}. \tag{3.11}$$

[5] In (3.8) and (3.9) $\nu_\tau^t\left(\mathrm{d}x_\tau^t | x_0^\tau\right)$ denotes the conditional probability measure on $\left(\Omega_\tau^t, \mathcal{F}_\tau^t\right)$.

We show later that the cocycle relation (3.10), together with the normalization relation (3.9), ensures the compatibility of the time dependent POV measures.

For the introduced time-dependent QSR we have the following collections of time dependent quantum instruments

$$\left\{\mathcal{N}_0^t(\cdot)[\cdot]\colon 0 < t \leqslant T\right\} \tag{3.12}$$

$$\mathcal{N}_0^t\left(B_0^t\right)[Y] = \int_{B_0^t} \left(V_0^t\left(x_0^t\right)\right)^* Y V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) \qquad \forall B_0^t \in \mathcal{F}_t \qquad \forall Y \in \mathcal{B}(\mathcal{H}) \tag{3.13}$$

and

$$\left\{\mathcal{M}_0^t(\cdot)[\cdot])\colon 0 < t \leqslant T\right\} \tag{3.14}$$

$$\mathcal{M}_0^t\left(B_0^t\right)[\kappa] = \int_{B_0^t} V_0^t\left(x_0^t\right) \kappa \left(V_0^t\left(x_0^t\right)\right)^* \nu_0^t\left(\mathrm{d}x_0^t\right) \qquad \forall B_0^t \in \mathcal{F}_t \qquad \forall \kappa \in \mathcal{T}(\mathcal{H}). \tag{3.15}$$

The corresponding collection of time-dependent POV measures and the family of time-dependent un-normalized posterior states are presented as

$$\left\{M_0^t(\cdot)\colon 0 < t \leqslant T\right\} \tag{3.16}$$

$$M_0^t\left(B_0^t\right) = \int_{B_0^t} \left(V_0^t\left(x_0^t\right)\right)^* V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) \qquad \forall B_0^t \in \mathcal{F}_t \tag{3.17}$$

and

$$\left\{\eta_{\mathrm{out}}^t(\cdot;\rho_0)\colon 0 < t \leqslant T\right\} \tag{3.18}$$

$$\eta_{\mathrm{out}}^t\left(x_0^t;\rho_0\right) = V_0^t\left(x_0^t\right) \rho_0 \left(V_0^t\left(x_0^t\right)\right)^* \qquad \forall x_0^t \in \otimes_0^t \tag{3.19}$$

respectively.

The collection of time-dependent output laws has the form

$$\left\{\pi_0^t(\cdot;\rho_0)\colon 0 < t \leqslant T\right\} \tag{3.20}$$

with

$$\pi_0^t\left(B_0^t;\rho_0\right) = \int_{B_0^t} \mathrm{tr}\left\{V_0^t\left(x_0^t\right) \rho_0 \left(V_0^t\left(x_0^t\right)\right)^*\right\} \nu_0^t\left(\mathrm{d}x_0^t\right) \qquad \forall B_0^t \in \mathcal{F}_t. \tag{3.21}$$

At any moment of time t and for any $B_0^t \in \mathcal{F}_t$ the normalized posterior states are given by

$$\rho^t\left(B_0^t;\rho_0\right) = \frac{\int_{B_0^t} \eta_{\mathrm{out}}^t\left(x_0^t;\rho_0\right) \nu_0^t\left(\mathrm{d}x_0^t\right)}{\pi_0^t\left(B_0^t;\rho_0\right)} = \frac{\mathcal{M}_0^t\left(B_0^t\right)[\rho_0]}{\pi_0^t\left(B_0^t;\rho_0\right)}. \tag{3.22}$$

In the following we shall also use the notation for the unconditional posterior state

$$\rho^t(\rho_0) \equiv \rho^t\left(\Omega_0^t;\rho_0\right) = \mathcal{M}_0^t\left(\Omega_0^t\right)[\rho_0] \tag{3.23}$$

satisfying the initial condition $\rho^t(\rho_0) \to \rho_0$ as $t \downarrow 0$ in the trace norm on $\mathcal{T}(\mathcal{H})$.

Due to the relations (3.4), (3.9) and (3.10), for any $t > \tau$ we have the martingale property

$$\int_{\Omega_\tau^t} \left(V_0^t\left(x_0^t\right)\right)^* V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right) = \left(V_0^\tau\left(x_0^\tau\right)\right)^* V_0^\tau\left(x_0^\tau\right) \tag{3.24}$$

from which it follows that the collection (3.16) of time-dependent POV measures is compatible in time, that is, for any $B_0^\tau \in \mathcal{F}_\tau$ we have

$$\begin{aligned} M_0^t\left(B_0^\tau\right) &= \int_{B_0^\tau} \left(V_0^t\left(x_0^t\right)\right)^* V_0^t\left(x_0^t\right) \nu_0^t\left(\mathrm{d}x_0^t\right) \\ &= M_0^\tau\left(B_0^\tau\right). \end{aligned} \tag{3.25}$$

For any $0 < \tau < t$ and any $Y \in \mathcal{B}(\mathcal{H})$, for the instruments from the collections (3.12) and (3.14) we have the following properties:

$$\mathcal{N}_0^t\left(\mathrm{d}x_0^t\right)[Y] = \mathcal{N}_0^\tau\left(\mathrm{d}x_0^\tau\right)\left[\mathcal{N}_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right)[Y]\right] \tag{3.26}$$

$$\mathcal{M}_0^t\left(\mathrm{d}x_0^t\right)[\kappa] = \mathcal{M}_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right)\left[\mathcal{M}_0^\tau\left(\mathrm{d}x_0^\tau\right)[\kappa]\right] \tag{3.27}$$

where we have introduced the notation

$$\mathcal{N}_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right)[Y] = \left(V_\tau^t\left(x_0^t\right)\right)^* Y V_\tau^t\left(x_0^t\right)\nu_\upsilon^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right) \qquad \forall Y \in \mathcal{B}(\mathcal{H}) \tag{3.28}$$

$$\mathcal{M}_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right)[\kappa] = V_\tau^t\left(x_0^\tau\right)\kappa\left(V_\tau^t\left(x_0^t\right)\right)^* \nu_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right) \qquad \forall \kappa \in \mathcal{T}(\mathcal{H}) \tag{3.29}$$

for instruments $\mathcal{N}_\tau^t\left(\,\cdot\,\middle|x_0^\tau\right)[\cdot]$ and $\mathcal{M}_\tau^t\left(\,\cdot\,\middle|x_0^\tau\right)[\cdot]$, which we call conditional.

Due to the properties (3.4)–(3.11) the collection $\left\{\mathcal{M}_0^t\left(\Omega_0^t\right)[\cdot]\colon 0 < t \leqslant T\right\}$ with

$$\mathcal{M}_0^t\left(\Omega_0^t\right)[\kappa] = \int_{\Omega_0^t} V_0^t\left(x_0^t\right)\kappa\left(V_0^t\left(x_0^t\right)\right)^* \nu_0^t\left(\mathrm{d}x_0^t\right) \qquad \forall \kappa \in \mathcal{T}(\mathcal{H}) \tag{3.30}$$

and the collection $\left\{\mathcal{M}_\tau^t\left(\Omega_\tau^t\middle|x_0^\tau\right)[\cdot]\colon 0 < \tau < t \leqslant T\right\}$ with

$$\mathcal{M}_\tau^t\left(\Omega_\tau^t \middle| x_0^\tau\right)[\kappa] = \int_{\Omega_s^t} V_\tau^t\left(x_0^t\right)\kappa\left(V_\tau^t\left(x_0^t\right)\right)^* \nu_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right) \qquad \forall \kappa \in \mathcal{T}(\mathcal{H}) \tag{3.31}$$

constitute families of time-dependent dynamical maps (cf, for example, [29]). We shall call the dynamical map, which we introduce by (3.31), conditional.

It follows also from (3.4)–(3.11) that for any $0 < \tau < t \leqslant T$ the time-dependent dynamical maps $\mathcal{M}_0^t\left(\Omega_0^t\right)[\cdot]$, $\mathcal{M}_\tau^t\left(\Omega_\tau^t\middle|x_0^\tau\right)[\cdot]$ are strongly continuous in t from the right with the following limits:

$$\lim_{t\downarrow 0}\left\|\mathcal{M}_0^t\left(\Omega_0^t\right)[\kappa] - \kappa\right\|_{\mathcal{T}(\mathcal{H})} = 0 \tag{3.32}$$

$$\lim_{t\downarrow \tau}\left\|\mathcal{M}_\tau^t\left(\Omega_\tau^t\middle|x_0^\tau\right)[\kappa] - \kappa\right\|_{\mathcal{T}(\mathcal{H})} = 0 \tag{3.33}$$

$\forall k \in \mathcal{T}(\mathcal{H})$, $\forall x_0^\tau \in \Omega_0^\tau$.

4. Posterior pure state trajectories in a Hilbert space

Together with an arbitrary pure initial state ψ_0, any given collection of quantum stochastic evolution operators $\left\{V_0^t(\cdot)\colon 0 < t \leqslant T\right\}$, with the properties specified in section 3, determines by

$$\phi\left(t\middle|x_0^t\right) = V_0^t\left(x_0^t\right)\psi_0 \tag{4.1}$$

a posterior pure state trajectory $\left\{\phi\left(\tau\middle|x_0^\tau\right)\right\}_{\tau\in(0,T]}$ in the Hilbert space $\otimes_{\tau\in(0,T]}\mathcal{H}$ conditioned by the continuously observed trajectory ${x_0}^T$ in the classical world.

Due to the specification of the time-dependent QSR, presented in (3.4)–(3.11), this trajectory is continuous in t from the right for $\forall t \in (0, T]$

$$\lim_{t\downarrow\tau}\left\|\phi\left(t\middle|x_0^t\right) - \phi\left(\tau\middle|x_0^\tau\right)\right\|_{\mathcal{H}} = 0. \tag{4.2}$$

Furthermore, $\phi\left(t\middle|x_0^t\right)$ satisfies the limit condition

$$\lim_{t\downarrow 0}\phi\left(t\middle|x_0^t\right) = \psi_0 \tag{4.3}$$

and, for any $0 < t \leqslant T$, the following normalization relation holds

$$\int_{\Omega_0^t}\left\|\phi\left(t\middle|x_0^t\right)\right\|^2 \nu\left(\mathrm{d}x_0^t\right) = 1. \tag{4.4}$$

According to QSA, $\left\{\phi\left(\tau \middle| x_0^\tau\right)\right\}_{\tau\in(0,t]}$ presents a *posterior pure state outcome under the continuous-time measurement*, which depends on the observed trajectory $x_0{}^{\mathrm{t}}$ in the classical world.

Thus, for the case of measurement continuous in time until the moment t, both the observed outcome x_0^t in the classical world and the posterior pure state outcome $\left\{\phi\left(\tau \middle| x_0^\tau\right)\right\}_{\tau\in(0,t]}$ in the Hilbert space $\otimes_{\tau\in(0,t]}\mathcal{H}$ are represented as trajectories.

Introduce also for any $\psi \in \mathcal{H}$ and any $s \leqslant t$ the notation

$$\Phi\left(t, s; x_0^t, \psi\right) = V_s^t\left(x_0^t\right)\psi. \tag{4.5}$$

Then from (3.9) it follows that for any $\psi \in \mathcal{H}$

$$\int_{\Omega_s^t} \left\|\Phi\left(t, s; x_0^t, \psi\right)\right\|^2 \nu\left(\mathrm{d}x_s^t \middle| x_0^s\right) = \|\psi\|^2 \tag{4.6}$$

and, due to the property (3.10), we have the following relation

$$\Phi\left(t; s; x_0^t, \Phi\left(s, \tau; x_0^s, \psi\right)\right) = \Phi\left(t, \tau; x_0^t, \psi\right) \tag{4.7}$$

valid ν-a.e. on Ω for any $t \in (0, T]$, $\tau \in [0, T]$, $s \in (0, T]$, $\tau \leqslant s \leqslant t$. In particular, since $\phi\left(s \middle| x_0^s\right) \equiv \Phi\left(s, 0; x_0^s, \psi_0\right)$ we can also write

$$\Phi\left(t; s; x_0^t, \phi\left(s \middle| x_0^s\right)\right) = \phi\left(t \middle| x_0^t\right). \tag{4.8}$$

If the initial state ρ_0 of a quantum system is pure, that is $\rho_0 = |\psi_0\rangle\langle\psi_0|$, then under the continuous-time direct measurement, described by the simple QSR, specified in section 3, at any moment $t \in (0, T]$ the probability (3.21) of the observed record $x_0^t = \{x_\tau\}_{\tau\in(0,T]}$ to belong to a subset $B_0^t \subseteq \Omega_0^t$ is given by

$$\pi_0^t\left(B_0^t; \rho_0\right) = \int_{B_0^t} \left\|\phi\left(t \middle| x_0^t\right)\right\|^2 \nu_0^t\left(\mathrm{d}x_0^t\right) \tag{4.9}$$

and, due to (4.6) and (4.8), the collection $\left\{\pi_0^t(\cdot; \rho_0) \colon t \in (0, T]\right\}$ of output laws is compatible in time.

The conditional posterior state (3.21) and the unconditional posterior state (3.23) are represented as

$$\rho^t\left(B_0^t; \rho_0\right) = \frac{\int_{B_0^t} \left|\phi\left(t \middle| x_0^t\right)\right\rangle\left\langle\phi\left(t \middle| x_0^t\right)\right| \nu_0^t\left(\mathrm{d}x_0^t\right)}{\pi_0^t\left(B_0^t; \rho_0\right)} \tag{4.10}$$

and

$$\rho^t(\rho_0) = \int_{\Omega_0^t} \left|\phi\left(t \middle| x_0^t\right)\right\rangle\left\langle\phi\left(t \middle| x_0^t\right)\right| \nu_0^t\left(\mathrm{d}x_0^t\right) \tag{4.11}$$

and, thus, correspond, respectively, to conditional and unconditional *statistical averaging* over the posterior pure state outcomes $\left|\phi\left(t \middle| x_0^t\right)\right\rangle\left\langle\phi\left(t \middle| x_0^t\right)\right|$ with respect to the input probability distribution $\nu_0^t(\cdot)$.

5. The case of Markov evolution

Consider now the special case of continuous-time measurement under which the quantum stochastic evolution operators $V_\tau^t\left(x_0^t\right)$ satisfy the following restriction

$$V_\tau^t\left(x_0^t\right) = V_\tau^t\left(x_\tau^t\right) \qquad \text{for all} \quad 0 \leqslant \tau < t \tag{5.1}$$

and the input probability scalar measure $\nu(\cdot)$ satisfies the relation

$$\nu\left(\mathrm{d}x_{\tau_1}^{\tau_2} \times \mathrm{d}x_{t_1}^{t_2}\right) = \nu\left(\mathrm{d}x_{\tau_1}^{\tau_2}\right)\nu\left(\mathrm{d}x_{t_1}^{t_2}\right) \tag{5.2}$$

for any $0 \leqslant \tau_1 < \tau_2 \leqslant t_1 < t_2 \leqslant T$.

In this special case, the conditional instruments (3.29) and (3.30) become unconditional in the sense that they do not depend on outcomes of measurements in the past:

$$\mathcal{N}_s^t\left(\mathrm{d}x_s^t \middle| x_0^s\right)[\cdot] = \mathcal{N}_s^t\left(\mathrm{d}x_s^t\right)[\cdot] \tag{5.3}$$

$$\mathcal{M}_s^t\left(\mathrm{d}x_s^t \middle| x_0^s\right)[\cdot] = \mathcal{M}_s^t\left(\mathrm{d}x_s^t\right)[\cdot] \tag{5.4}$$

and for any $0 < s < t \leqslant T$ we have

$$\mathcal{N}_0^t\left(B_0^s \times B_s^t\right) = \mathcal{N}_0^s\left(B_0^s\right) \circ \mathcal{N}_s^t\left(B_s^t\right) \tag{5.5}$$

$$\mathcal{M}_0^t\left(B_0^s \times B_s^t\right) = \mathcal{M}_s^t\left(B_s^t\right) \circ \mathcal{M}_0^s\left(B_0^s\right). \tag{5.6}$$

Thus, under the restrictions (5.1) and (5.2), the time-dependent collections of quantum instruments (3.12) and (3.14), with the properties (3.27) and (3.28), constitute the so-called instrumental processes with independent increments [4–6, 29, 30]. Notice that in the general QSA framework, considered in sections 3 and 4, the families of instruments do not generally satisfy the relations (5.5) and (5.6), which are, however, usually assumed to be valid for the description of continuous-time measurements in the frame of the operational approach (cf [30]).

The collection $\left\{\mathcal{M}_\tau^t\left(\Omega_\tau^t\right)[\cdot]\colon 0 \leqslant \tau < t \leqslant T\right\}$ where

$$\mathcal{M}_\tau^t\left(\Omega_\tau^t\right)[\cdot] = \int_{\Omega_\tau^t} V_\tau^t\left(x_\tau^t\right)[\cdot]\left(V_\tau^t\left(x_\tau^t\right)\right)^* \nu_\tau^t\left(\mathrm{d}x_\tau^t\right) \tag{5.7}$$

constitutes a family of time-dependent dynamical maps which, in contrast to (3.31), does not depend on outcomes in the past.

Under the restrictions (5.1) and (5.2) the unconditional posterior state $\rho^t(\rho_0)$, given by (3.23), has the following Markov property

$$\rho^t(\rho_0) = \mathcal{M}_s^t\left(\Omega_s^t\right)\left[\rho^s(\rho_0)\right] \qquad \forall 0 \leqslant s < t \tag{5.8}$$

In (5.8), we denote $\rho^0(\rho_0) := \lim_{s\downarrow 0}\rho^s(\rho_0) = \rho_0$ where the limit is in trace norm on $\mathcal{T}(\mathcal{H})$.

However, in contrast to the usual presentation of Markovian evolution of an open system (cf [29] and references cited therein) under the restrictions (5.1) and (5.2), the family of dynamical maps (5.7) does not, in general, represent a quantum dynamical semigroup.

Example. In recent years, the different stochastic calculus models of continuous-time quantum measurement, based on the introduction of linear (as well as nonlinear) stochastic differential equations for a process $\{\psi_t, t \in [0, \infty)\}$ with values in a complex separable Hilbert space $\mathcal{H}$, were intensively discussed in the mathematical and physical literature.

As we have already mentioned in the introduction, the type of stochastic equation used in all these presentations corresponds to the quantum filtering equation derived in [7–10], for the quantum stochastic calculus model of continuous-time indirect nondemolition measurements.

In the physical literature, in fact, only particular cases of such equations were considered. In the most general settings, the mathematical properties of this kind of stochastic model on a filtered probability space $(\Omega, \{F_t\}, F, P)$ were analysed in [2, 4–6, 27, 29, 30].

For the type of stochastic model in question it is postulated [6] that

- the (un-normalized) posterior state vector $\psi_t \in \mathcal{H}$ of the quantum system under continuous-time observation satisfies a stochastic differential equation of Ito's type

$$\mathrm{d}\psi_t = -K_t\psi_{t-}\,\mathrm{d}t + \sum L_{kt}\psi_{t-}\,\mathrm{d}W_{kt} + \int_{\mathcal{Y}} (J_t\psi_{t-})(y)\tilde{\Pi}(\mathrm{d}y, \mathrm{d}t) \tag{5.9}$$

with a nonrandom initial condition $\psi_0 = u \in \mathcal{H}$;

- the R^d-valued observed output process is defined to be

$$X_i(t) := \int_0^t c_i(s)\,\mathrm{d}s + \sum_{k=1}^{\infty}\int_0^t a_{ik}(s)\,\mathrm{d}W_{ks} + \int_{\mathcal{Y}\times(0,t]} \varphi(g(y;s))g_i(y;s)\Pi(\mathrm{d}y,\mathrm{d}s)$$
$$+ \int_{\mathcal{Y}\times(0,t]} \frac{g_i(y;s)}{1+|g_i(y;s)|^2}\tilde{\Pi}(\mathrm{d}y,\mathrm{d}t) \tag{5.10}$$

with the functions

$$\begin{aligned} c&: (0,\infty) \to R^d \\ a_{ik}&: (0,\infty) \to R \\ g&: \mathcal{Y}\times(0,\infty) \to R^d \\ \varphi(g)&: = \frac{\|g\|^2}{1+\|g\|^2}; \end{aligned} \tag{5.11}$$

- $i = 1,\ldots,d;\ k = 1,2,\ldots.$

The following assumptions are supposed to hold [6] for the stochastic model, defined by (5.9)–(5.11):

- For any $t \in (0,\infty)$ and any $k = 1,2,\ldots$ the operators $K_t \in \mathcal{B}(\mathcal{H})$, $L_{kt} \in \mathcal{B}(\mathcal{H})$, $J_t \in \mathcal{B}(\mathcal{H}, L^2(\mathcal{Y}, \nu(\cdot); \mathcal{H}))$;
- $K_t + K_t^* = \sum_{k=1}^{\infty} L_{kt}^* L_{kt} + J_t^*(I\otimes\gamma_t)J_t$ with γ_t being a bounded multiplication operator on the space $L^2(\mathcal{Y}, \nu(\cdot); \mathbb{C})$;
- The W_{kt} are independent Brownian motions;
- $\Pi(\mathrm{d}y, \mathrm{d}t)$ is an adapted Poisson point process on $\mathcal{Y}\times[0,\infty)$ of intensity $\gamma_t(y)v(\mathrm{d}y)\,\mathrm{d}t$ and increments independent of the past;
- $\tilde{\Pi}(\mathrm{d}y, \mathrm{d}t) = \Pi(\mathrm{d}y, \mathrm{d}t) - \gamma_t(y)v(\mathrm{d}y)\,\mathrm{d}t$ is the compensated Poisson process;
- $W = \{W_{kt}\}$ and $\Pi(\mathrm{d}y, \mathrm{d}t)$ are independent;

Due to these assumptions, under the law P the output process (5.10) is a process with independent increments. Let $\mathcal{E}_s^t$, $0 \leqslant s \leqslant t$ denote the σ-algebra generated by $X(r) - X(s)$, $r \in [s,t]$. Under a number of regularity conditions, it was proved in [6] that:

- The Cauchy problem for the equation (5.9), with a nonrandom initial condition $\psi_0 = u$ at time $t = 0$ has a unique (up to P-equivalence) solution $\psi_t = \Psi_t(0;\omega;u)$ and for any $t \geqslant 0$ the process $\|\psi_t\|^2$ is a positive martingale with

$$\mathrm{E}_P\left\{\|\psi_t\|^2\right\} = \|u\|^2. \tag{5.12}$$

- For any $t \geqslant s$ the solution $\Psi_t(s;\omega;\xi(\omega))$ of the equation (5.9) on the interval $(s,t]$ with the initial (at time s) random condition $\xi(\omega)$, where $\mathrm{E}_P\left\{\|\xi\|^2\right\} < \infty$, satisfies the relation

$$\mathrm{E}_P\left\{\|\Psi_t(s;\ \cdot\,;\xi)\|^2\right\} = \mathrm{E}_P\{\|\xi\|^2\} \tag{5.13}$$

and P-almost surely

$$\Psi_t(\tau;\omega;\Psi_\tau(s,\omega;u)) = \Psi_t(s;\omega;u) \qquad \forall\, t \geqslant \tau \geqslant s \tag{5.14}$$

- For any $u \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$ and any $B_s^t \in \mathcal{E}_s^t$ the equation

$$\left\langle u, \mathcal{N}_s^t\left(B_s^t\right)[A]u\right\rangle := \mathrm{E}_P\left\{1_{B_s^t}\langle\Psi_t(s;\ \cdot\,;u), A\Psi_t(s;\ \cdot\,;u)\rangle\right\} \tag{5.15}$$

defines a family of instruments $\left\{\mathcal{N}_s^t(\cdot)[\cdot]: 0 \leqslant s < t\right\}$ with the property (5.5).

Recalling the results of section 4, we see that the stochastic model of continuous-time quantum measurement, based on the introduction of the stochastic differential equation of type (5.9), corresponds to a special case of our presentation, where for any $\omega \in \Omega_s^t$ the quantum stochastic evolution operator is defined by the equation

$$\Psi_t(s;\omega;\psi) = V_s^t(\omega)\psi \qquad \forall\psi \in \mathcal{H}. \tag{5.16}$$

The unconditional posterior state of a quantum system in this stochastic model has the Markov property (5.8).

6. Measuring model of continuous-time direct quantum measurement

For any moment of time $t \in (0, T]$ let us construct, up to unitary and phase equivalence, a statistical realization, corresponding to the time-dependent QSR, specified in section 3. We shall refer to the resulting realization as the measuring model of continuous-time direct quantum measurement, corresponding to the (simple) QSR.

Let $\mathcal{H}(\nu) \equiv \mathcal{H}(\nu, N; \Omega_0^T)$ be the direct integral [12], induced by a probability scalar measure $\nu(\cdot)$ and dimension function $N(x_0^T)$ equal to identity ν-a.e. on Ω_0^T. For such a dimension function the direct integral $\mathcal{H}(\nu)$ is identical to $L_2(\Omega_0^T, \nu; \mathbb{C})$.

The relation

$$\left(X_0^T\left(B_0^T\right)\varphi_T\right)\left(x_0^T\right) = \chi_{B_0^T}\left(x_0^T\right)\varphi_T\left(x_0^T\right) \tag{6.1}$$

$\forall B_0^T \in \mathcal{F}$, $\forall\varphi_T \in \mathcal{H}(\nu)$, holding ν-a.e. on Ω_0^T, defines a simple projection-valued measure $X_0^T(\cdot)$: $\mathcal{F} \to \mathcal{B}(\mathcal{H}(\nu))$ of the type $\left[X_0^T(\cdot)\right] = [\nu(\cdot)]$ (cf [12, 36]). Here $\chi_{B_0^T}$ denotes the indicator function of a subset B_0^T.

Letting

$$X_\tau^t\left(B_\tau^t\right) = X_0^T\left(\Omega_0^\tau \times B_\tau^t \times \Omega_t^T\right) \tag{6.2}$$

the projection-valued measure $X_0^T(\cdot)$ defines by

$$\left(X_\tau^t\left(B_\tau^t\right)\varphi_T\right)\left(x_0^T\right) = \chi_{B_\tau^t}\left(x_\tau^t\right)\varphi_T\left(x_0^T\right) \qquad \forall B_\tau^t \in \mathcal{F}_\tau^t \quad \forall\varphi_T \in \mathcal{H}(\nu) \tag{6.3}$$

a collection $\left\{X_\tau^t(\cdot)\colon 0 \leqslant \tau < t \leqslant T\right\}$ of time-dependent, mutually commuting and compatible, projection-valued measures $X_\tau^t(\cdot)$ on the standard filtered Borel space $\left(\Omega_0^T, \{\mathcal{F}_t\}, \mathcal{F}\right)$ with values in $\mathcal{B}(\mathcal{H}(\nu))$, satisfying for any $0 \leqslant \tau < s < t \leqslant T$ the relation:

$$X_\tau^t\left(B_\tau^s \times B_s^t\right) = X_s^t\left(B_s^t\right)X_\tau^s\left(B_\tau^s\right) \tag{6.4}$$

$\forall B_s^t \in \mathcal{F}_s^t$, $\forall B_\tau^s \in \mathcal{F}_\tau^s$. In (6.3), x_τ^t is the restriction of x_0^T to the space Ω_τ^t. For any $t > \tau \geqslant 0$ the type $\left[X_\tau^t(\cdot)\right]$ is equal to $\left[\nu_\tau^t(\cdot)\right]$.

In the case considered, where $N(x_0^T) = 1$, ν-a.e. on Ω_0^T, a base of measurability (cf [12, 36]) consists of only one element e_T, defined, up to unitary equivalence, by the relation $\left|e_T\left(x_0^T\right)\right| = 1$, ν-a.e. on Ω_0^T. Since the measure $\nu(\cdot)$ is finite, $e_T \in \mathcal{H}(\nu)$ and is an element of maximum type for every projection-valued measure $X_\tau^t(\cdot)$:

$$\left\langle e_T, X_\tau^t(\cdot)e_T\right\rangle_{\mathcal{H}(\nu)} = \nu_\tau^t(\cdot). \tag{6.5}$$

Now, introduce the complex separable Hilbert space $\mathcal{K}(\nu) = \mathcal{H} \otimes \mathcal{H}(\nu)$.

Let $U_\nu(t, 0)$ be a unitary operator on $\mathcal{K}(\nu)$, strongly continuous in t from the right for $\forall t \in (0, T]$, satisfying the initial condition $s\text{-}\lim_{t\downarrow 0} U_\nu(t, 0) = I$ (strong limit) and such that for any vector $\psi \in \mathcal{H}$ the relation

$$(U_\nu(t,0)(\psi \otimes e_T))\left(x_0^T\right) = V_0^t\left(x_0^t\right)\psi \otimes e_T\left(x_0^T\right) \tag{6.6}$$

is valid ν_0^T-a.e. on Ω_0^T. The unitary operator $U_\nu(t, 0)$ is defined by the relation (6.6) up to unitary equivalence. The continuity conditions are required for the compatibility of the properties of $U_\nu(t, 0)$ with the properties of the quantum stochastic evolution operators $V_0^t\left(x_0^t\right)$, specified by (3.5)–(3.11).

The statistical realization

$$\left\{\mathcal{H}(\nu), |e_T\rangle\langle e_T|, X_0^t(\cdot), U_\nu(t, 0)\right\} \tag{6.7}$$

at any moment of time $t \in (0, T]$, presents on $\mathcal{H}(\nu)$ the invariant class $G(t)$ (cf [36]) of unitarily and phase equivalent separable statistical realizations, corresponding to the time-dependent QSR, specified in section 2.2.

For any $t \in (0, T]$, the representation of the instrument (3.13) through the elements of the statistical realization (6.7) is given by

$$\mathcal{N}_0^t\left(B_0^t\right)[Y] = \mathbf{E}_{|e_T\rangle\langle e_T|}\left[U_\nu^*(t, 0)\left(Y \otimes X_0^t\left(B_0^t\right)\right)U_\nu(t, 0)\right] \tag{6.8}$$

where for any statistical operator σ on $\mathcal{H}(\nu)$ the notation $\mathbf{E}_\sigma[\cdot]$ denotes the normal completely positive bounded linear map $\mathbf{E}_\sigma[\cdot]$: $\mathcal{B}(\mathcal{K}(\nu)) \rightarrow \mathcal{B}(\mathcal{H})$, such that for $\forall Q \in \mathcal{B}(\mathcal{K}(\nu))$ the relation

$$\operatorname{tr}\{\rho\mathbf{E}_\sigma[Q]\} = \operatorname{tr}\{(\rho \otimes \sigma)Q\} \tag{6.9}$$

is valid for any statistical operator ρ on $\mathcal{H}$ [42].

The family (3.16) of POV measures is represented by

$$M_0^t\left(B_0^t\right) = \mathbf{E}_{|e_T\rangle\langle e_T|}\left[U_\nu^*(t, 0)\left(I \otimes X_0^t\left(B_0^t\right)\right)U_\nu(t, 0)\right]. \tag{6.10}$$

Similar to (6.6), introduce also for any $t \geqslant \tau > 0$ the unitary operator $U_\nu(t, \tau)$, strongly continuous in t from the right, satisfying the relation $U_\nu(\tau, \tau) = I$ and such that for $\forall \psi \in \mathcal{H}$

$$(U_\nu(t, \tau)(\psi \otimes e_T))\left(x_0^T\right) = V_\tau^t\left(x_0^t\right)\psi \otimes e_T\left(x_0^T\right) \tag{6.11}$$

ν-a.e. on Ω_0^T. Then we have the following relation for the conditional instrument (3.28):

$$\int_{B_0^t} \mathcal{N}_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right)[Y]\nu_0^\tau\left(\mathrm{d}x_0^\tau\right) = \mathbf{E}_{|e_T\rangle\langle e_T|}\left[U_\nu^*(t, \tau)(Y \otimes X_0^t\left(B_0^t\right))U_\nu(t, \tau)\right] \qquad \forall Y \in \mathcal{B}(\mathcal{H}) \tag{6.12}$$

and, consequently, $\mathcal{N}_\tau^t\left(\mathrm{d}x_\tau^t \middle| x_0^\tau\right)[\cdot]$ is the Radon–Nikodym derivative with respect to $\nu_0^\tau(\cdot)$ of the instrument standing on the right-hand side of (6.12).

Due to (3.9), (6.5) and (6.12), for any $t \geqslant \tau \geqslant s > 0$ we have

$$\begin{aligned}\nu_0^s(\cdot) &= \mathbf{E}_{|e_T\rangle\langle e_T|}\left[U_\nu^*(t, \tau)\left(I \otimes X_0^s(\cdot)\right)U_\nu(t, \tau)\right] \\ &= \mathbf{E}_{|e_T\rangle\langle e_T|}\left[I \otimes X_0^s(\cdot)\right]\end{aligned} \tag{6.13}$$

and, therefore,

$$\mathbf{E}_{|e_T\rangle\langle e_T|}\left[U_\nu^*(t, \tau)\left[I \otimes X_0^s(\cdot), U_\nu(t, \tau)\right]\right] = 0 \tag{6.14}$$

where $T \geqslant t \geqslant \tau \geqslant s > 0$. From (6.14) it follows then that the family

$$\left\{U_\nu(t, \tau)\colon t \in (0, T];\ \tau \in [0, T];\ t \geqslant \tau;\ U_\nu(\tau, \tau)|_{\tau>0} = I;\ s\text{-}\lim_{t\downarrow 0} U_\nu(t, 0) = I\right\} \tag{6.15}$$

of unitary operators, strongly continuous in t from the right, satisfying (6.6) and (6.11), has the property:

$$\left[I \otimes X_0^s(\cdot), U_\nu(t, \tau)\right](\psi \otimes e_T) = 0 \tag{6.16}$$

$\forall \psi \in \mathcal{H}; \forall t \geqslant \tau \geqslant s > 0$.

Let $\mathcal{H}_R$ be a complex separable Hilbert space isometrically isomorphic to $\mathcal{H}(\nu)$ by a unitary transform R, that is $\mathcal{H}_R = RH(\nu)$. The relation $P_R^{(\tau,t]}(\cdot) = RX_\tau^t(\cdot)R^{-1}$ defines the family

$$\left\{P_R^{(\tau,t]}(\cdot)\colon T \geqslant t > \tau \geqslant 0\right\} \tag{6.17}$$

of mutually commuting, compatible projection-valued measures $P_R^{(\tau,t]}(\cdot)\colon \mathcal{F}_\tau^t \to \mathcal{B}(\mathcal{H}_R)$ of the type $\left[\nu_\tau^t(\cdot)\right]$, satisfying

$$P_R^{(\tau,t]}\left(B_s^t \times B_\tau^s\right) = P_R^{(\tau,s]}\left(B_s^t\right) P_R^{(s,t]}\left(B_\tau^s\right) \tag{6.18}$$

$\forall B_s^t \in \mathcal{F}_s^t$, $\forall B_\tau^s \in \mathcal{F}_\tau^s$; $\forall t > s > \tau \geqslant 0$.

Let

$$\left\{U_R(t,\tau)\colon t \in (0,T];\ \tau \in [0,T];\ t \geqslant \tau;\ U_R(\tau,\tau)|_{\tau>0} = I;\ s\text{-}\lim_{t\downarrow 0} U_R(t,0) = I\right\} \tag{6.19}$$

be the family of unitary operators on $\mathcal{K}_R = (I \otimes R)\mathcal{K}(\nu) = \mathcal{H} \otimes \mathcal{H}_R$, corresponding to $U_\nu(t,\tau)$ on $\mathcal{K}(\nu)$. Then

$$U_R(t,\tau) = (I \otimes R)U_\nu(t,\tau)(I \otimes R^{-1}). \tag{6.20}$$

Denote $f_R = Re_T$. From (6.17) and (6.20) it follows that for any $\psi \in \mathcal{H}$

$$\left[U_R(t,\tau), I \otimes P_R^{(0,s]}(\cdot)\right](\psi \otimes f_R) = 0 \qquad \forall t \geqslant \tau \geqslant s > 0. \tag{6.21}$$

Furthermore, for any $t > \tau \geqslant 0$ and any $\psi \in \mathcal{H}$:

$$\left(I \otimes P_R^{(0,t]}\left(\mathrm{d}x_0^t\right)\right)U_R(t,\tau)(\psi \otimes f_R) = \left(V_\tau^t\left(x_0^t\right) \otimes P_R^{(0,t]}\left(\mathrm{d}x_0^t\right)\right)(\psi \otimes f_R) \tag{6.22}$$

$$U_R(t,\tau)(\psi \otimes f_R) = \int_{\Omega_0^t} \left(V_\tau^t\left(x_0^t\right) \otimes P_R^{(0,t]}\left(\mathrm{d}x_0^t\right)\right)(\psi \otimes f_R) \tag{6.23}$$

$$\left\langle f_R, P_R^{(\tau,t]}(\cdot) f_R\right\rangle_{\mathcal{H}_R} = \nu_\tau^t(\cdot) \tag{6.24}$$

where the relation (6.22) should be understood in the infinitesimal sense. For the description of the most general case of continuous-time nondemolition measurement the relations (6.22)–(6.24) were first introduced in [35].

From (6.22)–(6.24), we have

$$\mathbf{E}_{|f_R\rangle\langle f_R|}[U_R(t,0)] = \int_{\Omega_0^t} V_0^t\left(x_0^t\right)\nu_0^t\left(\mathrm{d}x_0^t\right) \tag{6.25}$$

$$\mathbf{E}_{|f_R\rangle\langle f_R|}\left[\left(I \otimes P_R^{(0,t]}\left(\mathrm{d}x_0^t\right)\right)U_R(t,\tau)\right] = V_\tau^t\left(x_0^t\right)\nu_0^t\left(\mathrm{d}x_0^t\right). \tag{6.26}$$

From (3.10), (6.21) and (6.22) it follows that for any $t \geqslant s \geqslant \tau > 0$ and $\forall\psi \in \mathcal{H}$ the unitary operators $U_R(t,\tau)$ (and, hence, also the unitary operators $U_\nu(t,\tau)$) satisfy the following relation

$$U_R(t,\tau)(\psi \otimes f_R) = U_R(t,s)U_R(s,\tau)(\psi \otimes f_R) \tag{6.27}$$

which we call a cocycle property with respect to the vector $f_R \in \mathcal{H}_R$.

The statistical realization

$$\left\{\mathcal{H}_R, |f_R\rangle\langle f_R|, P_R^{(0,t]}(\cdot), U_R(t,0)\right\} \tag{6.28}$$

is unitarily equivalent to the statistical realization (6.7) and at any moment $t \in (0,T]$ presents, in general, the invariant class $G(t)$ of unitarily and phase equivalent statistical realizations, corresponding to the time-dependent QSR, specified in section 3.

We shall call the 4-tuple

$$\left\{\mathcal{H}_R, |\varphi_R\rangle\langle\varphi_R|, P_R^{(0,T]}(\cdot), \{U_R(t,\tau), : 0 \leqslant \tau < t \leqslant T\}\right\} \tag{6.29}$$

represented by a simple projection-valued measure $P_R^{(0,T]}(\cdot)\colon \Omega_0^T \to \mathcal{B}(\mathcal{H}_R)$ and a family of unitary operators (6.19), with properties (6.21) and (6.22)–(6.27), respectively, the *measuring model of continuous-time direct quantum measurement*, *corresponding to a simple time-dependent QSR*.

7. Scheme for continuous-time indirect nondemolition measurement

Building on [35], we now consider the scheme for continuous-time indirect measurement presented in [7–10]. This type of measurement implies that indirect information about the quantum system S is obtained via a direct measurement upon another quantum system, say R (with a Hilbert space $\mathcal{H}_R$), entangled with S. The unitary evolution of the compound system (S plus R) on the complex separable Hilbert space $\mathcal{K}_R = \mathcal{H} \otimes \mathcal{H}_R$ is described in the frame of the Hamiltonian approach, while the description of a direct measurement upon the quantum system R from the point of view of QSA should be based on the introduction of a corresponding QSR.

However, up to the present moment, the consideration in the physical and the mathematical literature of continuous-time indirect observation on the system S has been given, in fact, only for a special case, where the POV measure of the continuous-time direct measurement upon the quantum system R is presented by the joint spectral measure of a family of self-adjoint, time-dependent operators $\{Q_H(t): t \in (0, T]\}$ on $\mathcal{K}_R$, mutually commuting

$$[Q_H(t), Q_H(\tau)] = 0 \qquad \forall t, \tau \in (0, T] \tag{7.1}$$

and corresponding in the Heisenberg picture to some observable of the quantum system R.

A von Neumann observable $Q_H(t)$, $t \in (0, T]$, satisfying the condition (7.1) is usually termed nondemolition [29, 44, 35] or self-nondemolition (cf [7–10] and references therein).

However, as was pointed out in [7–10], the condition (7.1) alone does not ensure the existence, at any moment of time $t \in (0, T]$, of an instrument (with respect to the quantum system S) which describes, via (2.3), conditional expectations of any von Neumann S-system observable Z under continuous-time indirect measurement and, consequently, allows the introduction of the family of posterior states (cf (2.8)).

That is why, in [7–10], along with the condition (7.1) there was also introduced an additional condition, specified below by (7.2). These two conditions are required to represent *continuous-time indirect nondemolition measurement* and are announced in [7–10] as *'principles of continuous in time nondemolition observation'*.

Let $\{U(t, \tau): t, \tau \in [0, T]\}$ be the cocycle of unitary operators, describing the evolution of the compound system (S plus R) in the interaction picture, induced by the free dynamics of the system R (cf, for example, [35]). Then, according to the definition given in [7–10], under continuous-time indirect nondemolition measurement:

- there must exist the nondemolition observable $Q_H(t) = U^*(t, 0)(I \otimes Q_R(t))U(t, 0)$, corresponding to some free dynamical observable $Q_R(t)$ of the system R;
- at any moment of time $t \in [0, T]$ any von Neumann S-system observable $Z_H(t) = U^*(t, 0)(Z \otimes I)U(t, 0)$, $Z \in \mathcal{B}(\mathcal{H})$, where $Z = Z^*$, must commute with the observables $Q_H(s)$ at all previous moments of time:

$$[Z_H(t), Q_H(s)] = 0 \qquad \forall t \geqslant s \geqslant 0. \tag{7.2}$$

Suppose, for simplicity, that for the family of self-adjoint, mutually commuting operators $\{Q_H(t), t \in (0, T]\}$ its joint spectrum [12] coincides with $\Omega_0^T = D(0, T]$.

We are now in a position to prove the following statement:

Proposition. *In the most general case, that is, without specifying a concrete nondemolition measurement model, the simultaneous fulfilment of conditions (7.1) and (7.2) is equivalent to the following:*

- *the family of self-adjoint operators* $\{Q_R(t) \colon t \in (0, T]\}$ *is a family of mutually commuting operators such that their joint spectral projection-valued measure* $P_R^{(0,T]}(\cdot) \colon \Omega_0^T \to \mathcal{B}(\mathcal{H}_R)$*, for any* $T \geqslant t \geqslant \tau > 0$*, satisfies the commutativity relation*

$$\left[U(t, \tau), I \otimes P_R^{(0,\tau]}(\cdot)\right] = 0 \tag{7.3}$$

where $P_R^{(0,t]}\left(B_0^t\right) = P_R^{(0,T]}\left(B_0^t \times \Omega_t^T\right)$ for $\forall B_0^t \in \mathcal{F}_t$. *(Note that*

$$[U(t, \tau), I \otimes Q_R(s)] = 0 \tag{7.4}$$

$\forall t \geqslant \tau \geqslant s > 0$, *presents an equivalent formulation of the relation (7.3)).*

Proof. Let (7.1) and (7.2) be satisfied. Then from the condition (7.1) it follows (cf [12]) that there exists a joint projection-valued measure $P_H^{(0,T]}(\cdot) \colon \Omega_0^T \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_R)$ such that for any $t \in (0, T]$

$$Q_H(t) = \int_{\Omega_0^T} x_t P_H^{(0,T]}\left(\mathrm{d}x_0^T\right) = \int_{\Omega_0^t} x_t P_H^{(0,t]}\left(\mathrm{d}x_0^t\right) \tag{7.5}$$

where the projection-valued measure $P_H^{(0,t]}\left(B_0^t\right) = P_H^{(0,T]}\left(B_0^t \times \Omega_t^T\right)$.

For any $Z \in \mathcal{B}(\mathcal{H})$, $Z = Z^*$, the commutativity relation (7.2) is then equivalent to

$$\left[Z_H(t), P_H^{(0,t]}(\cdot)\right] = 0 \tag{7.6}$$

and, hence, to

$$\left[Z \otimes I, U(t, 0) P_H^{(0,t]}(\cdot) U^*(t, 0)\right] = 0. \tag{7.7}$$

Since (7.7) is valid for any von Neumann S-system observable $Z \in \mathcal{B}(\mathcal{H})$, by the commutation theorem of von Neumann algebras the projection-valued measure $U(t, 0) P_H^{(0,t]}(\cdot) U^*(t, 0)$ must have the form:

$$U(t, 0) P_H^{(0,t]}(\cdot) U^*(t, 0) = I \otimes P_R^{(0,t]}(\cdot). \tag{7.8}$$

From (7.8), it follows that the relations

$$\begin{aligned} P_H^{(0,\tau]}(\cdot) &= U^*(\tau, 0)\big(I \otimes P_R^{(0,\tau]}(\cdot)\big) U(\tau, 0) \\ &= U^*(t, 0)\big(I \otimes P_R^{(0,\tau]}(\cdot)\big) U(t, 0) \end{aligned} \tag{7.9}$$

are valid for any $t \geqslant \tau > 0$. The relation (7.3) follows from (7.9) trivially.

Furthermore, due to $Q_H(t) = U^*(t, 0)(I \otimes Q_R(t)) U(t, 0)$ and the relations (7.8) and (7.9), for any $t > 0$ we have the following representation:

$$Q_R(t) = \int_{\Omega_0^T} x_t P_R^{(0,T]}\big(\mathrm{d}x_0^T\big) = \int_{\Omega_0^t} x_t P_R^{(0,t]}\big(\mathrm{d}x_0^t\big). \tag{7.10}$$

Consequently, the von Neumann observable $Q_R(t)$, $t \in (0, T]$ is also nondemolition.

The proof of the converse statement is straightforward. □

We would like to emphasize here that although the conditions (7.1) and (7.2) do not imply any concrete measurement model, the consideration of continuous-time indirect nondemolition measurement (cf [7–10] and references therein), leading to the derivation of the quantum filtering equation, was presented only in the frame of quantum stochastic calculus. The measurement model of quantum stochastic calculus is essentially Markovian. That is why, as already pointed out in the introduction and section 5, under the scheme of continuous-time indirect nondemolition measurement, the quantum filtering equation [7–10], as well as its further analogues [4–6, 27, 29, 30], correspond to quite special stochastic models, which are Markovian.

8. The scheme for continuous-time indirect nondemolition measurement as a special measuring model of continuous-time measurement

In this section we show that:

- For the general measuring model (6.29) of continuous-time direct quantum measurement, there exists a uniquely determined family $\{Q_R(t), t \in (0, T]\}$ of mutually commuting (and hence nondemolition) self-adjoint operators on $\mathcal{H}$, defined on a common domain D. For any $0 < \tau \leqslant t \leqslant T$ the joint spectral measure of these operators satisfies the relation
$$\left[U(t, \tau), I \otimes P_R^{(0,\tau]}(\cdot)\right](\psi \otimes f_R) = 0. \tag{8.1}$$
Since the condition (7.3) is only sufficient for the relation (8.1) to be valid and since, in contrast to the model of 'continuous-time indirect nondemolition measurement', the unitary operators $U(t, \tau), 0 \leqslant \tau < t \leqslant T$ in (6.29) are strongly continuous in t only from the right, the model of continuous-time 'indirect nondemolition measurement' represents only a special case of the measuring model (6.29) of continuous-time observation of a quantum system.
- For the most general (that is, not only in the frame of quantum stochastic calculus) model of continuous-time nondemolition measurement with initial state of the system R being pure, under some further technical (for simplicity) restrictions, specified below, there exists the uniquely defined simple time-dependent QSR, introduced in section 2 and satisfying properties (3.4)–(3.11).

Consider the first point: let $P_R^{(0,T]}$ be the projection-valued measure of a measuring model (6.29). Introduce the system of self-adjoint operators $\{Q_R(t): t \in (0, T]\}$ given by
$$Q_R(t) = \int_{\Omega_0^T} x_t P_R^{(0,T]}\left(\mathrm{d}x_0^T\right) = \int_{\Omega_0^t} x_t P_R^{(0,t]}\left(\mathrm{d}x_0^t\right). \tag{8.2}$$
These operators are mutually commuting (cf [12]) with a common domain
$$D = \left\{ f \in \mathcal{H}_R\colon \int_{\Omega_0^T} (x_t)^2 \nu_f\left(\mathrm{d}x_0^T\right) < \infty, \forall t \in (0, T] \right\} \tag{8.3}$$
where the probability scalar measure $\nu_f\left(\mathrm{d}x_0^T\right) = \left\langle f, P_R^{(0,T]}\left(\mathrm{d}x_0^T\right) f\right\rangle$ on Ω_0^T. The relation (8.1) corresponds then to (6.21).

Let us now prove the second point: come back to the notation of section 7. Let f_R be the initial state of the quantum system R and let the conditions (7.1) and (7.2) be satisfied. Suppose also, for simplicity, that the joint spectrum of the family of nondemolition observables $\{Q_H(t): t \in (0, T]\}$ coincides with $\Omega_0^T = D(0, T]$.

Then, according to the consideration in section 7, $\{Q_R(t): t \in (0, T]\}$ must be a family of self-adjoint mutually commuting observables with the joint projection-valued measure $P_R^{(0,T]}(\cdot)\colon \Omega_0^T \to \mathcal{B}(\mathcal{H}_R)$, satisfying the relation (7.3).

The relation
$$\left\langle f_{R,} P_R^{(0,T]}(\cdot) f_R \right\rangle = \nu_0^T(\cdot) \tag{8.4}$$
determines the probability scalar measure $\nu_0^T(\cdot)$ on the filtered space $\left(\Omega_0^T, \{\mathcal{F}_t\}, \mathcal{F}\right)$. For simplicity, suppose that $P_R^{(0,T]}(\cdot)$ is simple. The family of quantum stochastic evolution operators $V_\tau^t(\cdot)\colon \Omega_0^t \to \mathcal{B}(\mathcal{H}), \forall t, \tau \in (0, T]\}$, is then introduced similarly to (6.11) (cf also [36]) and has the properties (3.5)–(3.11).

9. Concluding remarks

A full description and classification, in terms of invariants, of all possible integral representations of any given quantum instrument were presented in [36] where the interpretation of the derived mathematical results for the quantum measurement theory was also proposed and discussed. In the present paper we consider the further development of the general quantum stochastic approach, introduced in [36, 37], for the description, in the most general case, of statistical and stochastic aspects under continuous-time measurement.

Specifying, in general, the time-wise properties of a quantum stochastic evolution operator, describing the stochastic evolution of an open quantum system subjected to continuous-time observation, we introduce the notion of a conditional quantum instrument and discuss the properties of the families of time-dependent quantum instruments that describe a continuous-time measurement. We present also the time-wise specifications of compatibles in time outcome laws and the formulae for the conditional and unconditional posterior states.

Further, we define, in the most general case and without assuming any Markov property, the notion of the posterior pure state trajectories in a Hilbert space and present their (compatible in time) probabilistic description. The restrictions, under which the stochastic evolution of a continuously observed quantum system is Markovian, are also specified.

We construct a 'canonical' measuring model of a continuous-time observation of an open quantum system and prove that, formally, the scheme for continuous-time indirect nondemolition measurement represents a special case of this model.

Acknowledgments

The present work was carried out in the framework of MaPhySto (Centre for Mathematics and Stochastics), which is funded by the Danish National Research Foundation.

References

[1] Barchielli A 1986 Measurement theory and stochastic differential equations in quantum mechanics *Phys. Rev.* A **34** 1642

[2] Barchielli A and Belavkin V P 1991 Measurements continuous in time and *a posteriori* states in quantum mechanics *J. Phys. A: Math. Gen.* **24** 1495

[3] Barchielli A 1993 On the quantum theory of measurements in continuous time *Rep. Math. Phys.* **33** 21

[4] Barchielli A and Holevo A S 1995 Constructing quantum measurement processes via classical stochastic calculus *Stochastic Process. Appl.* **58** 293

[5] Barchielli A and Paganoni A M 1996 A note on a formula of the Levy–Khinchin type in quantum probability *Nagoya Math. J.* **141** 29

[6] Barchielli A 1998 On stochastic differential equations and semigroups of probability operators in quantum probability *Stochastic Process. Appl.* **73** 69

[7] Belavkin V P 1992 Quantum continual measurements and *a posteriori* collapse on CCR *Commun. Math. Phys.* **146** 611

[8] Belavkin V P 1992 Quantum stochastic calculus and quantum nonlinear filtering *J. Multivariate. Anal.* **42** 171

[9] Belavkin V P 1994 Nondemolition principle of quantum measurement theory *Found. Phys.* **24** 685

[10] Belavkin V P 2001 Quantum noise, bits and jumps: uncertainties, decoherence, measurements and filtering *Progress in Quantum Electronics* vol **25** (Amsterdam: Elsevier) pp 1–53

[11] Braginsky V B and Khalili F Y 1992 *Quantum Measurement* (Cambridge: Cambridge University Press)

[12] Birman M Sh and Solomyak M Z 1987 *Spectral Theory of Self-Adjoint Operators in Hilbert Space* (Dordrecht: Reidel)

[13] Carmichael H J 1993 *An Open System Approach in Quantum Optics (Lecture Notes in Physics 18)* (New York: Springer)

[14] Dalibard J, Castin Y and Mølmer K 1992 Wave function approach to dissipative processes in quantum optics *Phys. Rev. Lett.* **68** 580

[15] Davies E B and Levis J T 1970 An operational approach to quantum probability *Commun. Math. Phys.* **17** 239

[16] Davies E B 1976 *Quantum Theory of Open Systems* (London: Academic)

[17] Diosi L 1988 Continuous quantum measurement and Ito formalism *Phys. Lett.* A **129** 419

[18] Diosi L and Strunz W T 1997 The non-Markovian stochastic Schrödinger equation for open systems *Phys. Lett.* A **235** 569

[19] Doherty A C, Tan S M, Parkins A S and Walls D F 1999 State determination in continuous measurement *Phys. Rev.* A **60** 2380

[20] Duan L M, Cirac J I, Zoller P and Polzik E S 2000 Quantum communication between atomic ensembles using coherent light *Phys. Rev. Lett.* **85** 5643

[21] Furusawa A, Sorensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E S 1998 Unconditional quantum teleportation *Science* **282** 706

[22] Ghirardi G C, Pearle P and Rimini A 1990 Markov processes in Hilbert space and continuous sponataneous localization of systems of identical particles *Phys. Rev.* A **42** 78

[23] Gisin N 1984 Quantum measurement and stochastic processes *Phys. Rev. Lett.* **52** 1657

[24] Gisin N and Percival I C 1992 The quantum-state diffusion model applied to open systems *J. Phys. A: Math. Gen.* **25** 5677

[25] Goetsch P and Graham R 1994 Linear stochastic wave equation for continuously measurement quantum systems *Phys. Rev.* A **50** 5242

[26] Holevo A S 1980 *Probabilistic and Statistical Aspects of Quantum Theory* (Moscow: Nauka); (English transl. 1982 (Amsterdam: North-Holland))

[27] Holevo A S 1996 Stochastic differential equations in Hilbert space and quantum Markovian evolutions *Probability Theory and Mathematical Statistics, Proc. 7th Japan–Russia Symp (Tokyo, 26–30 July 1995)* ed S Watanabe, M Fukushima, Yu V Prohorov and A N Shiryaev (Singapore: World Scientific) pp 122–31

[28] Holevo A S 1998 Radon–Nikodym derivatives of quantum instruments *J. Math. Phys.* **39** 1373

[29] Holevo A S 2001 *Statistical Structure of Quantum Theory* (Berlin: Springer)

[30] Holevo A S 2001 Levy processes and continuous quantum measurements *Levy Processes—Theory and Applications* ed O E Barndorff-Nielsen, T Mikocsh and S Resnick (Boston: Birkhäuser) pp 225–39

[31] Hood C J, Lynn T W, Doherty A C, Parkins A S and Kimble H J 2000 The atom-cavity microscope. Single atoms bound in orbit by single photons *Science* **287** 1447

[32] Kraus K 1983 *States, Effects and Operations: Fundamental Notions of Quantum Theory* (Berlin: Springer)

[33] Lindblad G 1976 On the generators of quantum dynamical semigroups *Commun. Math. Phys.* **48** 119

[34] Loubenets E R and Åberg C 1994 Quantum stochastic calculus and nondemolition measurements (Notes on a recent approach to quantum measurements) *Research report* N 94-28, ITP, Göteborg, Sweden

[35] Loubenets E 1999 The quantum stochastic evolution of an open system under continuous-time nondemolition measurement *Research report* 1999-45, MaPhySto, University of Aarhus, Denmark

[36] Loubenets E R 2001 Quantum stochastic approach to the description of quantum measurements *J. Phys. A: Math. Gen.* **34** 7639

[37] Loubenets E R 2001 Quantum stochastics. The new approach to the description of quantum measurements *Quantum Probability and White Noise Analysis, vol 12, Proc. Workshop Foundations of Probability and Physics (Vaxjo, Sweden, 25 Nov–1 Dec 2000)* ed A Khrennikov (Singapore: World Scientific)

[38] Mensky M B 1993 *Continuous Quantum Measurements and Path Integrals* (Bristol: Institute of Physics Publishing)

[39] Mølmer K and Castin Y 1996 Monte Carlo wavefunctions *Coherence and Quantum Optics* **7** 193

[40] Von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press)

[41] Ou Z Y, Pereira S F, Kimble H J and Peng K C 1992 Realization of the Einstein–Podolsky–Rosen paradox for continuous-variables *Phys. Rev. Lett.* **68** 3663

[42] Ozawa M 1984 Quantum measuring processes of continuous observables *J. Math. Phys.* **25** 79

[43] Ozawa M 1985 Conditional probability and a posteriori states in quantum mechanics *Publ. RIMS, Kyoto Univ.* **21** 279

[44] Peres A 1993 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)

[45] Peres A 2000 Classical intervention in quantum systems. The measuring process *Phys. Rev.* A **61** 022116

[46] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics* (New York: Academic)
[47] Strunz W T 1996 Linear quantum state diffusion for non-Markovian open quantum systems *Phys. Lett.* A **224** 25
[48] Strunz W T, Diosi L and Gisin N 1999 Open systems with non-Markovian quantum trajectories *Phys. Rev. Lett.* **82** 1801
[49] Strunz W T, Diosi L, Gisin N and Yu T 1999 Quantum trajectories for Brownian motion *Phys. Rev. Lett.* **83** 4909
[50] Wiseman H M 1996 Quantum trajectories and quantum measurement theory *Quantum Semiclass. Opt.* **8** 205
[51] Wiseman H M 1999 Adaptive quantum measurements *Proc. Workshop on Stochastics and Quantum Physics Miscellanea* vol 16 (MaPhySto, University of Aarhus) pp 89–93